

# Counterterms for Linear Divergences in Real-Time Classical Gauge Theories at High Temperature

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Real-time classical  $SU(N)$  gauge theories at non-zero temperature contain linear divergences. We introduce counterterms for these divergences in the equations of motion in the continuum and on the lattice. These counterterms can be given in terms of auxiliary fields that satisfy local equations of motion. We present a lattice model with 6+1D auxiliary fields that for IR-sensitive quantities yields cut-off independent results to leading order in the coupling. Also an approximation with 5+1D auxiliary fields is discussed.

## I. INTRODUCTION

In recent years there has been a considerable interest in the calculation of the Chern-Simons diffusion rate in the symmetric phase of the electroweak theory. By the chiral anomaly this diffusion rate is related to the rate of baryon number non-conservation [1] and therefore it is an important parameter in electroweak baryogenesis scenarios; see for reviews e.g. [2,3]. It was originally suggested by Grigoriev and Rubakov [4] that such infrared sensitive quantities could be determined from a classical theory. Their idea was that field configurations at momenta  $k \ll T$  are expected to behave classically:

$$n(k) = \frac{1}{e^{\beta k} - 1} \sim \frac{1}{\beta k} = n_{cl}(k) . \quad (1)$$

Since the diffusion of Chern-Simons number is dominated by soft fields with typical momenta  $k \sim g^2 T$ , it was expected that it could be calculated in a classical theory [5,6]. Later, it was argued by Arnold, Son and Yaffe [7] that this picture is not complete, because soft field configurations are sensitive to (particle-like) thermal fluctuations with hard momenta  $k \sim T$ , that are not correctly reproduced by the classical theory. These hard modes introduce Landau damping and slow down the dynamics of nearly static magnetic fields at momenta  $k \sim g^2 T$ . Including these hard modes leads to an estimate of a typical time-scale  $t \sim (g^4 T)^{-1}$  of these fields. Bödeker [8] showed that also semi-hard modes at momentum scale  $gT$  contribute to leading order, and a even logarithmic correction  $t \sim (g^4 T \log \frac{1}{g})^{-1}$  to the time-scale is obtained. This logarithmic correction has been calculated in various approaches [9–15].

To calculate the Chern-Simons diffusion rate in an (effective) classical theory with cut-off  $\Lambda$ , the corrections of the hard modes and semi-hard modes have to be incorporated. For the semi-hard modes this requires that the cut-off is large enough,  $\Lambda \gg gT$ . To include the hard modes basically two approaches can be taken. One approach is that one uses the classical hard modes to mimic the effect of the hard modes in the quantum theory [16]. These classical hard modes introduce linear divergences in the theory [7,17] and if one takes the (continuum) cut-off  $\Lambda \sim T$ , the classical hard modes generate the (classical) hard thermal loops and with a proper matching this incorporates Landau damping in the correct way.

A different approach is to include the hard thermal loops in the classical theory [18–22]. This can be done in an economic way by including an induced source in the equations of motion for the fields and to solve for the dynamics of the induced source in the presence of the classical fields. In that case one also has the problem of the linear divergences introduced by the classical hard modes. In the continuum these can be accounted for by adjusting the strength of the induced source [20,23].

On the lattice both these approaches have the problem that the linear divergences are not rotationally invariant [17]. Therefore, the classical hard modes cannot correctly mimic the continuum hard thermal loop effects, like Landau damping, nor can these be accounted for by adjusting the strength of the (continuum) induced source. The lattice spacing dependence of the classical theory has been studied in [24,25]

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The aim of this paper is to investigate the inclusion of counterterms for the linear divergences on a lattice. Via a continuum model (section II) and a lattice model (section III) suitable for perturbative calculations only, we arrive at two models that may be used for non-perturbative calculations on a lattice.

In section IV A the first non-perturbative model is discussed, it contains the exact counterterms for the linear divergences on a lattice with lattice spacing  $a_L$ . The counterterms are generated by 6+1D auxiliary fields that satisfy local equations of motion. The finite renormalization is chosen such that the theory is matched to a quantum lattice theory, with a smaller lattice spacing  $a_S$ . For very small couplings  $g$ , the lattice spacing  $a_S$  can be chosen to be small compared to the inverse temperature and continuum results can be obtained.

The second model is presented in section IV B, it contains the counterterms in the approximation that the external momenta of a divergent diagram are strongly space-like ( $p_0 \ll |\vec{p}|$ ). This model requires 5+1D auxiliary fields and  $g$  is unrestricted.

## II. CONTINUUM

In this section we review the HTL equations of motion in the continuum and show how the counterterms for the linear classical divergences may be obtained. It is convenient to use the local formulation of the HTL equation of Blaizot and Iancu [26]. They showed that the HTL equation is equivalent to a set of linearized kinetic equations. The first equation describes classical gauge fields in the presence of an induced source

$$[D_\mu, F^{\mu\nu}(x)] = j_{\text{ind}}^\nu(x) , \quad (2)$$

with the covariant derivative  $D_\mu = \partial_\mu + igA_\mu$  and field strength  $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$ . The induced current can be written as an integral over four-momentum  $K$  of the source density  $j^\nu(K, x)$

$$j_{\text{ind}}^\nu(x) = g \int \frac{d^4 K}{(2\pi)^4} j^\nu(x, K) . \quad (3)$$

The source density satisfies the equation

$$[K_\mu D^\mu, j^\nu(x, K)] = 2gN K^\nu K^\rho F_{\rho\sigma}(x) \partial_K^\sigma \Delta(K) , \quad (4)$$

with  $N$  the number of colors and  $\Delta(K)$  the free symmetric two-point correlation function of the fields that have been integrated out to obtain equations (2-4). The factor 2 accounts for the spin degrees of freedom. Equations (2-4) are equivalent to the HTL's calculated with propagator  $\Delta(K)$ .

The kinetic equations (2-4) are derived under the assumptions that the theory is weakly coupled and that  $x$  is a slowly varying variable,  $\partial \ll K \sim T$ . Since we are interested in the divergences of the theory the loop momenta can be taken of the order of the cut-off  $k \sim \Lambda$  which is much larger than the fixed external momentum  $Q \sim \partial$ .

In the continuum the two-point function takes the form

$$\Delta(K) = [\Theta(K_0) - \Theta(-K_0)] \delta(K^2) N(K_0) , \quad (5)$$

with  $N(K_0)$  the equilibrium distribution function of the fluctuations. The  $\delta$ -function in (5) forces the fluctuations on-shell with dispersion relation  $K^0 = \pm k$ ,  $k = |\vec{k}|$ . Inserting the propagator (5) in (4), the solution for the source density can be written as

$$j^\nu(x, K) = 2N K^\nu \delta(K^2) \left[ \Theta(K_0) \delta N(x, \vec{k}) - \Theta(-K_0) \delta N(x, -\vec{k}) \right] . \quad (6)$$

The function  $\delta N(x, \vec{k})$  may be interpreted as a particle distribution function, because it satisfies the kinetic equation

$$\left[ v^\mu D_\mu, \delta N(x, \vec{k}) \right] = g v^\rho F_{\rho 0} N'(k) , \quad (7)$$

with the velocity  $v^\mu = (1, \vec{v})$  and  $\vec{v} = \vec{k}/k$  and  $N'(k)$  the derivative of the equilibrium distribution function to the energy.

The induced source is proportional to the deviation from equilibrium of the density of particles in the plasma

$$j_{\text{ind}}^\nu(x) = g \int \frac{d^3 k}{(2\pi)^3} v^\nu \delta N(x, \vec{k}) , \quad (8)$$

with  $g$  the gauge coupling.

Our task is to specify what we mean by the particles of the plasma. If we would consider QED, the particles would be the fermions and the equilibrium particle distribution function would be the Fermi-Dirac distribution function. In the  $SU(N)$  gauge theory that we consider here, the particles are gauge field excitations. If the gauge fields in (2) and (7) would be interpreted as mean fields then the particles can be identified with the thermal fluctuations and  $N(k)$  would be the Bose-Einstein distribution function  $n_{BE}(k)$  [26]. However we consider the equations (2-7) as the equations of motion of a classical statistical theory, where still a thermal average over the initial fields has to be taken. The particles in this case are the quantum fluctuations with distribution function

$$N(k) = n_{BE}(k) - T/k, \quad (9)$$

as can be derived from separating off the classical fields from the path integration and calculating the two-point function of the remaining fields [27] which appears in (4).

The equations (2-7) may be simplified by introducing the field

$$W(x, \vec{v}) = -\delta N(x, \vec{k}) (gN'(k))^{-1}. \quad (10)$$

It follows from (7) that  $W$  is independent of the energy  $k$ , assuming this is consistent with the initial data. The kinetic equation (7) in terms of this field reads

$$[v^\mu D_\mu, W(x, \vec{v})] = \vec{v} \cdot \vec{E}(x). \quad (11)$$

The induced current (8) can be written as

$$j_{\text{ind}}^\nu(x) = -2g^2 N \int \frac{dk}{2\pi^2} k^2 N'(k) \int \frac{d\Omega}{4\pi} v^\nu W(x, \vec{v}), \quad (12)$$

where the angular integration is over the direction of  $\vec{v}$ . As usual the  $k$ -integration decouples from the angular integration in the HTL approximation.

Due to the subtraction in  $N(k)$  (9), the integration over  $k$  introduces a linear divergence. This divergence acts as a counterterm for the linear divergences in the classical theory [28]. Let us argue why this is correct. The argument used in [28] is one of consistency: since the full quantum correlation functions cannot contain classical linear divergences, the quantum corrections to the classical equations should provide the correct counterterms. Another argument is that integrating out the classical fluctuations around a given background field generates the classical induced source in the effective equations for the background field. Since the classical induced source (in the HTL approximation) is precisely what is subtracted in (12), the equations of motion for the background field do not contain linear divergent terms. The logarithmic divergences of the classical theory are more complicated and are not considered here.

If we use dimensional regularization, the subtraction does not contribute, since linear divergences are set equal to zero. We then have

$$j_{\text{ind}}^\nu(x) = 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} v^\nu W(x, \vec{v}), \quad (13)$$

with the plasmon frequency  $\omega_{\text{pl}}^2 = g^2 N T^2 / 9$ .

As a different regularization scheme that makes the linear divergence explicit, we may restrict the subtraction to momenta  $k < \Lambda$ :

$$N_\Lambda(k) = n_{BE}(k) - \frac{T}{k} \Theta(\Lambda - k). \quad (14)$$

The result of the  $k$ -integration in the induced source yields then

$$-2g^2 N \int \frac{dk}{2\pi^2} k^2 N'_\Lambda(k) = 3\omega_{\text{pl}}^2 - \frac{2}{\pi^2} g^2 N \Lambda T, \quad (15)$$

where we see an explicit linear divergence. The inclusion of a linear dependence on the cut-off in the plasmon frequency as in (15) was proposed in [20,23]. The derivation presented here and in more detail in [28] shows how such a cut-off dependence arises in a continuum theory as a consequence of the subtraction in the distribution function (9) for the quantum fluctuations.

From the point of view of renormalization the situation is quite remarkable: there is an infinite set of linearly divergent diagrams that are all non-local (or momentum-dependent) but the renormalized equations of motion can be brought in a local form and the linear divergence is introduced in just one parameter. Note also that the usual HTL's are just a finite renormalization.

### III. PERTURBATIVE RENORMALIZATION ON A LATTICE

Before turning to the HTL equations of motion, we shortly review the static classical theory on a lattice, as far as the linear divergences are concerned. The only linear divergence in the static theory occurs in the Debye mass. A counterterm for this divergence may be introduced in the mass of the temporal gauge field [29,30]

$$\delta m^2 = m_D^2 - m_{\text{cl}}^2, \quad (16)$$

with the continuum HTL contribution

$$m_D^2 = -2g^2 N \int \frac{d^3 k}{(2\pi)^3} n'(k) = \frac{1}{3} g^2 N T^2, \quad (17)$$

and the classical mass (for a simple cubic lattice with lattice spacing  $a$ )

$$m_{\text{cl}}^2 = -2g^2 N \int \frac{d^3 p}{(2\pi)^3} n'(\Omega_{\vec{p}}) \approx 0.51 g^2 N T a^{-1}, \quad (18)$$

where  $\vec{p}$  is restricted to the first Brillouin zone  $|p_i| \leq \pi/a$  and the energy  $\Omega_{\vec{p}}$  is

$$\Omega_{\vec{p}}^2 = \frac{4}{a^2} \left[ \sin^2 \left( \frac{p_x a}{2} \right) + \sin^2 \left( \frac{p_y a}{2} \right) + \sin^2 \left( \frac{p_z a}{2} \right) \right]. \quad (19)$$

The mass (18) is the linear divergent contribution to the Debye mass on the lattice, its subtraction in (16) ensures that no linear divergences are present in the static theory with the mass counterterm included. The continuum HTL contribution (17) to the counterterm mass (16) provides the finite renormalization, which is chosen such the  $a \rightarrow 0$  limit yields the continuum result.

We want to extend this approach to the real-time classical theory. We consider again the the equation for the gauge fields (2), but with a lattice regularization; time is continuous but space is a simple cubic lattice with lattice spacing  $a$ . Similar to the mass counterterm (16) in the static case, we want the source to contain a continuum HTL contribution with a classical lattice contribution subtracted. To this end, we introduce two particle densities  $\delta N(x, \vec{k})$  and  $\delta N_{\text{ct}}(x, \vec{p})$  for particles with energies  $E_{\vec{k}} = |\vec{k}|$  and  $\Omega_{\vec{p}}$  respectively. The idea is that the particle density  $\delta N_{\text{ct}}$  generates the counterterms for the linear divergences and  $\delta N$  generates the correct finite renormalization, the “good” HTL contributions. Then these particle densities satisfy the equations

$$\left[ v^\mu D_\mu, \delta N(x, \vec{k}) \right] = g v^\rho F_{\rho 0}(x) n'_{BE}(k), \quad (20)$$

$$[v_{\text{lat}}^\mu D_\mu, \delta N_{\text{ct}}(x, \vec{p})] = g v_{\text{lat}}^\rho F_{\rho 0}(x) n'_{\text{cl}}(\Omega_{\vec{p}}). \quad (21)$$

Here and in the following, we persist in using a continuum notation although we use a lattice regularization for the UV-divergences. The kinetic equation for  $\delta N(x, \vec{k})$  is eq.(7) with the Bose-Einstein distribution function as equilibrium distribution function. In the kinetic equation for  $\delta N_{\text{ct}}(x, \vec{p})$  the velocity on the lattice is  $v_{\text{lat}}^\mu = (1, \vec{v}_{\text{lat}})$  with [16]

$$v_{\text{lat}}^i = \partial_{p_i} \Omega_{\vec{p}} = \frac{1}{a \Omega_{\vec{p}}} \sin(ap_i), \quad (22)$$

and  $|\vec{v}_{\text{lat}}| \neq 1$  in general. The induced current in the classical equations of motion for the gauge fields then read

$$j_{\text{ind}}^\mu(x) = 2gN \int \frac{d^3 k}{(2\pi)^3} v^\mu \delta N(x, \vec{k}) - 2gN \int \frac{d^3 p}{(2\pi)^3} v_{\text{lat}}^\mu \delta N_{\text{ct}}(x, \vec{p}). \quad (23)$$

Here the integration over  $\vec{p}$  is restricted to the first Brillouin zone  $|p_i| < \pi/a$ .

As in the continuum, a field  $W(x, \vec{v})$  may be defined that satisfies (the lattice version of) equation (11). Since the lattice velocity (22) is not restricted to the speed of light, we have to allow for general velocities  $\vec{v}$  in (11). The induced current (23) reads

$$j_{\text{ind}}^\nu(x) = 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} v^\nu W(x, \vec{v}) - 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} v_{\text{lat}}^\nu W(x, \vec{v}_{\text{lat}}). \quad (24)$$

with the dimensionless quantities  $\hat{p}_i = ap_i$ ,  $\hat{\Omega}_{\vec{p}} = a\Omega_{\vec{p}}$  and the integration restricted to  $|\hat{p}_i| < \pi$ . The first term on the r.h.s. of (24) is the continuum contribution for which the  $k$ -integration decouples, as in (12), and has been performed.

In the second term on the r.h.s. of (24) the integration cannot be simplified since the velocity not only depends on the direction of the momentum  $\vec{p}$ , but also on its magnitude. For the calculation of the continuum contribution to the induced current a field  $W(x, \vec{v})$  depending on the direction of  $\vec{v}$  only is sufficient, however the lattice contribution requires fields that depend also on the magnitude of the velocity  $|\vec{v}_{\text{lat}}| < 1$ . In section IV B we will study the question if, for the calculation of the Chern-Simons diffusion rate, we may approximate the induced current with fields that only depend on the direction of the velocity.

We note that the induced current (24) is covariantly conserved  $[D_\mu, j_{\text{ind}}^\mu] = 0$ .

Just as the usual HTL equations, the equations (2), (11) and (24) (or equivalently (2), (20), (21) and (23)) define a perturbation theory. Taking retarded initial conditions the retarded propagator (and higher-order retarded vertex functions) can be obtained, as in [26]. The classical KMS condition then fixes the entire propagator, including its thermal part [31]. Using perturbation theory we may verify that also the time-dependent counterterms are correct, we calculate the retarded propagator to one-loop order. In a general gauge it takes the form

$$D_{\text{cl}}^{\mu\nu}(Q) = [g^{\mu\nu}Q^2 - Q^\mu Q^\nu + F^\mu F^\nu + \Pi_{\text{cl}}^{\mu\nu}(Q) + \delta\Pi_{\text{ind}}^{\mu\nu}(Q)]^{-1}, \quad (25)$$

with  $F^\mu$  the gauge fixing vector and  $\Pi_{\text{cl}}^{\mu\nu}$  the classical self-energy and  $\delta\Pi_{\text{ct}}^{\mu\nu}$  the counterterm self-energy introduced in the induced source (24). The classical self-energy to one-loop order reads [17,24]

$$\Pi_{\text{cl}}^{\mu\nu}(Q) = 2g^2 Na^{-1} \int \frac{d^3\hat{p}}{(2\pi)^3} n'_{\text{cl}}(\Omega_{\vec{p}}) \left[ -\delta^{\mu 0} \delta^{\nu 0} + \frac{v_{\text{lat}}^\mu v_{\text{lat}}^\nu q_0}{q_0 + i\epsilon - \vec{v}_{\text{lat}} \cdot \vec{q}} \right], \quad (26)$$

at this order the classical self-energy contains no contribution from the induced source. The linearized induced source

$$j_{\text{ind}}^\mu(x) = - \int d^4x' \delta\Pi_{\text{ind}}^{\mu\nu}(x, x') A_\nu(x') \quad (27)$$

defines the retarded self energy

$$\delta\Pi_{\text{ind}}^{\mu\nu}(Q) = \Pi_{\text{HTL}}^{\mu\nu}(Q) - \Pi_{\text{cl}}^{\mu\nu}(Q), \quad (28)$$

with the continuum HTL self-energy

$$\Pi_{\text{HTL}}^{\mu\nu}(Q) = 3\omega_{\text{pl}}^2 \left[ -\delta^{\mu 0} \delta^{\nu 0} + \int \frac{d\Omega}{4\pi} \frac{v^\mu v^\nu}{q_0 + i\epsilon - \vec{v} \cdot \vec{q}} \right]. \quad (29)$$

Inserting the counterterm self-energy (28) in the propagator (25), we see that the linear divergent classical self-energy and the linear divergent part of the counterterm self-energy cancel. The resulting self-energy in the propagator (25) is the correct (continuum) HTL self-energy. Finally we note that in the static limit the self-energy (28) reduces to the counterterm mass (16), as it should.

Unfortunately the system (2),(11), (24) is unsuitable for numerical implementation [32]. This can be seen from the conserved energy of the system

$$E = \int d^3x \frac{1}{2} \left[ (\vec{E}^b)^2 + (\vec{B}^b)^2 + 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} W^b(x, \vec{v}) W^b(x, \vec{v}) - 2g^2 N T a^{-1} \int \frac{d^3\hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} W^b(x, \vec{v}_{\text{lat}}) W^b(x, \vec{v}_{\text{lat}}) \right], \quad (30)$$

with  $\vec{B}$  the chromo-magnetic field and  $b$  the adjoint index. The energy is unbounded from below and this means that the system is unstable. In an ideal world the effect of the counterterm particle density is precisely compensated by the hard modes of the classical gauge fields. however in practice the evolution of the particle density and the hard modes will differ, which means that after some time the (wrong) effect of the counterterm particle density is no longer compensated by the hard modes, and the fields will (exponentially) blow up.

## IV. TWO MODELS WITH A BOUNDED ENERGY

### A. Model with lattice dispersion relation

The goal is to obtain a model that is defined on the lattice, that is stable and that can be used to calculate IR-sensitive real-time properties of a non-abelian plasma without linear divergences. Such a model should meet the

following three requirements:

- 1) In the small lattice spacing limit the continuum HTL equations of motion should be obtained.
- 2) Counterterms for the linear divergences (on the lattice) should be included.
- 3) The energy must be bounded from below.

The model considered in the previous section failed to have bounded energy. To obtain a model with a bounded energy one can consider a model where the modes inducing the finite renormalization have the same dispersion relation as the counterterm modes. In this section we focus on a model where both the counterterm modes and the modes generating the finite renormalization satisfy a lattice dispersion relation. The other possibility of enforcing the continuum dispersion relation on the counterterm modes is considered in the next section.

To obtain HTL equations where the both types of modes satisfy a lattice dispersion relation, we do not match to a continuum quantum theory as in the previous section, but to a quantum theory on the lattice, with a (small) lattice spacing  $a_S$ . The HTL equations (in the  $A_0 = 0$  gauge) may then be written as

$$[D_\mu, F^{\mu\nu}(x)] = j_{\text{ind}}^\nu(x) = 2gN \int \frac{d^3\hat{p}}{(2\pi)^3} v_{\text{lat}}^\nu \delta\tilde{N}(x, \hat{p}) , \quad (31)$$

$$\partial_t \delta\tilde{N}(x, \hat{p}) - [\vec{v}_{\text{lat}} \cdot \vec{D}, \delta\tilde{N}(x, \hat{p})] = -g\vec{v}_{\text{lat}} \cdot \vec{E}(x) \partial_{\hat{\Omega}_{\hat{p}}} \tilde{N}(\hat{\Omega}_{\hat{p}}) , \quad (32)$$

with  $x = (t, \vec{x})$ , where the time  $t$  is continuous and the position  $\vec{x}$  is an element of a cubic lattice with (large) lattice spacing  $a_L$ . The dimensionless momentum  $\hat{p}$  is restricted to the first Brillouin zone  $|\hat{p}_i| < \pi$ , the dimensionless energy  $\hat{\Omega}_{\hat{p}} = 2\sqrt{\sum_i \sin^2(\hat{p}_i/2)}$  and the velocity  $v_{\text{lat}}^i = \partial_{\hat{p}_i} \hat{\Omega}_{\hat{p}}$ .

The lattice spacing has been scaled out of the above equations and enters only in the equilibrium distribution function  $\tilde{N}$ . The distribution function  $\tilde{N}$  should contain a contribution that, after solving (32), generates the quantum HTL source and a contribution that generates the counterterms for the classical divergences. The important step is now to allow for different lattice spacings  $a_L, a_S$  in the the different parts of the equilibrium distribution function

$$\tilde{N}(\hat{\Omega}_{\hat{p}}) = a_S^{-2} n_{BE}^S(\hat{\Omega}_{\hat{p}}) - a_L^{-2} n_{\text{cl}}^L(\hat{\Omega}_{\hat{p}}) , \quad (33)$$

with

$$\begin{aligned} n_{BE}^S(\hat{\Omega}_{\hat{p}}) &= \frac{1}{e^{\hat{\Omega}_{\hat{p}}/(a_S T)} - 1} , \\ n_{\text{cl}}^L(\hat{\Omega}_{\hat{p}}) &= \frac{T a_L}{\hat{\Omega}_{\hat{p}}} , \end{aligned} \quad (34)$$

with  $T$  the temperature of the system.

To see that the model (31) and (32) contains the counterterms for the linear divergences it is useful to introduce the field

$$\tilde{W}(x, \hat{p}) = \delta\tilde{N}(x, \hat{p}) / \left( -g\tilde{N}'(\hat{\Omega}_{\hat{p}}) \right) , \quad (35)$$

which satisfies the equation

$$\partial_t \tilde{W}(x, \hat{p}) - [\vec{v}_{\text{lat}} \cdot \vec{D}, \tilde{W}(x, \hat{p})] = \vec{v}_{\text{lat}} \cdot \vec{E}(x) . \quad (36)$$

The source can be split into a part generating the finite quantum HTL source and a part subtracting the linear divergent classical source

$$j_{\text{ind}}^\nu = j_{\text{fin}}^\nu - j_{\text{ct}}^\nu . \quad (37)$$

In terms of the field  $\tilde{W}$  these sources read

$$j_{\text{fin}}^\nu = 2g^2 N \int \frac{d^3 p_S}{(2\pi)^3} v_{\text{lat}}^\nu n'_{BE}(\Omega_S) \tilde{W}(x, \vec{p}_S a_S) , \quad (38)$$

$$j_{\text{ct}}^\nu = 2g^2 N \int \frac{d^3 p_L}{(2\pi)^3} v_{\text{lat}}^\nu n'_{\text{cl}}(\Omega_L) \tilde{W}(x, \vec{p}_L a_L) , \quad (39)$$

with  $\vec{p}_S = a_S^{-1} \hat{p}$ ,  $\Omega_S = a_S^{-1} \hat{\Omega}_{\hat{p}}$  and similar for  $\vec{p}_L, \Omega_L$ . Both sources (38) and (39) are covariantly conserved.

Written in dimensionfull quantities we recognize the source  $j_{\text{ct}}$  (39) as the classical HTL source on a lattice with lattice spacing  $a_L$ . The difference with the perturbative model of the previous section is the choice of the finite renormalization. The source  $j_{\text{fin}}$  (38) is the quantum HTL source on a lattice with lattice spacing  $a_S$ . To extract continuum results from this model we should require  $a_S^{-1} \gg T$ . Also  $a_L$  cannot be too large, since the relevant field configurations for the sphaleron rate have size  $(g^2 T)^{-1}$ , we should at least require  $a_L^{-1} \gg g^2 T$ . However as Bödeker [8] has shown modes of spatial size  $(gT)^{-1}$  give corrections of  $\mathcal{O}(1)$ , to take these corrections into account requires a smaller lattice spacing  $a_L^{-1} \gg gT$ .

To ensure the stability of the model (31) and (32) we demand that the energy,

$$E = \int d^3x \frac{1}{2} \left[ (\vec{E}^b)^2 + (\vec{B}^b)^2 - 2N \int \frac{d^3\hat{p}}{(2\pi)^3} \delta\tilde{N}^b(x, \hat{p}) \delta\tilde{N}^b(x, \hat{p}) / \tilde{N}'(\hat{\Omega}_{\hat{p}}) \right], \quad (40)$$

is bounded from below. This leads to the requirement

$$-\tilde{N}'(\hat{\Omega}_{\hat{p}}) > 0. \quad (41)$$

For  $\hat{p} = 0$ , this requirement implies  $a_S < a_L$ , which is in accordance with the general idea that the classical theory is matched to a quantum theory with a smaller lattice spacing.

The function  $-\tilde{N}'(\hat{\Omega}_{\hat{p}})$ , with  $a_S < a_L$ , decreases from plus infinity at  $\hat{\Omega}_{\hat{p}} = 0$ , to its minimum below zero, after which it increases and asymptotically reaches zero. The maximum value of the dimensionless energy is  $\hat{\Omega}_{\hat{p}} = 2\sqrt{3}$ . Demanding that

$$-\tilde{N}'(2\sqrt{3}) > 0, \quad (42)$$

together with  $a_S < a_L$  is sufficient for (41) to hold for any  $\hat{p}$ . In this way, we obtain a maximum value for  $a_S^{-1}$  given the ratio  $a_L/a_S$ . In table I some results are listed. We see that in order to obtain continuum-like HTL contributions, the ratio  $a_L/a_S$  should be very (exponentially) large.

Since we want  $a_L^{-1} \gg gT$ , the coupling coupling  $g$  should be chosen extremely small. For instance, if we fix  $a_S^{-1} = 2.59T$ , then stability requires  $a_L/a_S \geq 100$ , so  $a_L^{-1} \leq 2.59 \cdot 10^{-2}T$  and  $g \ll 2.59 \cdot 10^{-2}$ .

TABLE I. The maximum value of  $a_S^{-1}/T$  given the ration  $a_L/a_S$  from the requirement that the energy is bounded from below.

$a_L/a_S$	1.1	1.5	2	5	10	20	25	50	100	1000
$\max(a_S^{-1})/T$	0.31	0.64	0.86	1.36	1.68	1.97	2.06	2.33	2.59	3.42

To complete the model we specify the average over the initial fields, that has to be taken to calculate classical correlation functions. The initial conditions are

$$\begin{aligned} \vec{E}(t_{\text{in}}, \vec{x}) &= \vec{E}_{\text{in}}(\vec{x}), \\ \vec{A}(t_{\text{in}}, \vec{x}) &= \vec{A}_{\text{in}}(\vec{x}), \\ \delta N(t_{\text{in}}, \vec{x}, \hat{p}) &= \delta N_{\text{in}}(\vec{x}, \hat{p}). \end{aligned} \quad (43)$$

Following [20,33], we include the auxiliary field in the average over initial fields

$$Z = \int D E_{\text{in}} D A_{\text{in}} D \delta N_{\text{in}} \delta(G_{\text{in}}) \exp(-\beta E), \quad (44)$$

with  $G_{\text{in}} = 0$  Gauss' law at the initial time. We have verified that the phase space measure is invariant under time evolution.

We may conclude that the model given by (32), (43), (44) describes a real-time quantum theory on lattice to leading order in  $g$  without cut-off dependence. For very small coupling the lattice spacing  $a_S$  of the quantum theory can be taken small compared to the inverse temperature and the model describes a quantum continuum theory. It can be used to calculate IR-sensitive quantities without lattice spacing dependence. Consider for instance the Chern-Simons diffusion rate, which in the small coupling limit takes the form [8]

$$\Gamma = \left[ \kappa_1 \log \frac{1}{g} + \kappa_2 \right] g^{10} T^4. \quad (45)$$

A calculation of  $\Gamma$  with the model given by (32), (43), (44) at fixed lattice spacing  $a_S$ , gives an  $a_L$ -independent result for both coefficients  $\kappa_1$  and  $\kappa_2$ , for  $a_L^{-1} \gg gT$ .

## B. Model with a continuum dispersion relation

The other approach that we want to investigate is a model where we enforce the continuum dispersion relation on the counterterm modes. Such a model has the advantage that instead of a 6+1D field  $\delta N$  a 5+1D auxiliary field  $W(x, \hat{v}_{\text{lat}})$ , that depends only on the direction of the velocity  $\hat{v}_{\text{lat}} = \vec{v}_{\text{lat}}/|\vec{v}_{\text{lat}}|$ , can be used. The counterterms that we obtain in this model are not exact, but for the calculation of the Chern-Simons diffusion rate the model provides a reasonable approximation.

The model that we consider is given by the replacement of the induced source (12) by the expression

$$j_{\text{app}}^\nu(x) = 3\omega_{\text{pl}}^2 \int \frac{d\Omega}{4\pi} v^\nu W(x, \vec{v}) - 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}| \tilde{v}_{\text{lat}}^\nu W(x, \hat{v}_{\text{lat}}), \quad (46)$$

with  $\tilde{v}_{\text{lat}}^\nu = (1, \hat{v}_{\text{lat}})$ . We choose this expression since it reproduces the induced vector current for a field configuration with  $W(x, \hat{v}_{\text{lat}}) = W(x, \vec{v}_{\text{lat}})$ , and the vector current is essential in the dynamics of the soft fields. The density is then determined by requiring current conservation  $[D_\mu, j_{\text{app}}^\mu] = 0$ . As a consequence the induced density  $j_{\text{ind}}^0$  in (24) is not correctly reproduced by the density  $j_{\text{app}}^0$ . This can be easily understood, changing the velocity of the particles and requiring current conservation either the vector current or the density can remain unaltered. The expression (46) is the lattice equivalent of the approximation for the induced source in [20].

We may also write (46) as

$$j_{\text{app}}^\nu(x) = \int \frac{d\Omega}{4\pi} m^2(\vec{v}) v^\nu W(x, \vec{v}), \quad (47)$$

with the velocity dependent mass

$$m^2(\vec{v}) = 3\omega_{\text{pl}}^2 - 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}| \delta^S(\vec{v} - \hat{v}_{\text{lat}}). \quad (48)$$

The second term contains a linear divergence in the direction  $\vec{v} = (1, 1, 1)/\sqrt{3}$  [32] and logarithmic divergences in directions  $\vec{v} = (1, 1, s)/\sqrt{2+s^2}$  with  $-1 < s < 1$  (and directions related by symmetry). Therefore the mass and the energy are not strictly positive. To obtain a bounded energy some averaging over the direction of the velocity  $\vec{v}$  should be performed. This can be achieved by expanding the field  $W(x, \vec{v})$  in spherical harmonics

$$W(x, \vec{v}) = \sum_{lm} W_{lm}(x) Y_{lm}(\vec{v}), \quad (49)$$

and keeping a finite number terms. The induced source can then be written as

$$j_{\text{ind}}^\nu(x) = \sum_{lm} a_{lm}^\nu W_{lm}(x), \quad (50)$$

with coefficients

$$a_{lm}^\nu = \int \frac{d\Omega}{4\pi} m^2(\vec{v}) v^\nu Y_{lm}(\vec{v}). \quad (51)$$

Given the lattice spacing  $a$ , the requirement that the energy is bounded from below puts an upper bound  $l_{\text{max}}$  on allowed values of  $l$ . It was found in [22] that the Chern-Simons diffusion rate is insensitive to  $l_{\text{max}}$  for even  $l_{\text{max}}$ . In the following we will therefore focus on the approximation made in (46).

As was already mentioned the approximation (46) changes the charge density. For instance for the coefficient  $a_{00}^0$ , we have

$$a_{00}^0 = m_D^2 - 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}|. \quad (52)$$

Comparing (52) with (18), we see that the expression (46) does not correctly reproduce the counterterm for the Debye mass. This implies that the current is not suitable to describe the behaviour of fields at length scale  $(gT)^{-1}$ .

To see whether the approximation (46) is valid for fields at the length scale  $(g^2 T)^{-1}$ , we consider the spatial components of the counterterm self-energy generated by the source (46) (for  $l_{\text{max}} \rightarrow \infty$ )



$$\Pi_{\text{ct}}^{ij}(q_0, \vec{q}) = \Pi_{HTL}^{ij}(q_0, \vec{q}) - \Pi_{\text{app}}^{ij}(q_0, \vec{q}) , \quad (53)$$

with

$$\Pi_{\text{app}}^{ij}(q_0, \vec{q}) = 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}| \frac{\hat{v}_{\text{lat}}^i \hat{v}_{\text{lat}}^j q_0}{q_0 + i\epsilon - \vec{v}_{\text{lat}} \cdot \vec{q}} , \quad (54)$$

which should be compared with the classical self-energy (26).

It is important to realize that the relevant fields for the Chern-Simons diffusion rate we are interested in, have typical momenta of order  $q_0 \sim g^4 T$ ,  $q \sim g^2 T$ . For the gauge fields relevant for the Chern-Simons diffusion rate  $q_0 \ll |\vec{q}|$  and neglecting  $q_0$  in the denominator of the counterterm (54) and the classical self-energy (26), we note that these are equal and that they cancel. For these fields the effective theory is finite and reproduces the HTL contributions.

However it was realized by Bödeker that interactions between semi-hard and soft fields give corrections to the dynamics of the soft fields that are not suppressed by powers of  $g$ . On the contrary, even  $\log(1/g)$  enhanced contributions arise. The counterterms in the approximated source (46) and the classical HTL's do not cancel for the semi-hard modes (with momenta  $q_0, q \sim gT$ ), therefore the semi-hard modes are sensitive to the cut-off  $a^{-1}$ .

The leading log contribution arises from the IR-sensitive part of the contribution of the semi-hard modes with momenta  $k_0 \ll k \sim \mu$  and  $\mu \sim g^2 T$  an IR cut-off. For these momenta the approximation is correct to leading order, therefore a calculation of the Chern-Simons diffusion rate with approximation (46) produces the correct leading log contribution, the coefficient  $\kappa_1$  in (45) is independent of the lattice spacing.

The  $\mathcal{O}(1)$  correction from the semi-hard modes does depend on the cut-off. An estimate of the cut-off dependence can be obtained from a comparison of the classical HTL self-energy (26) with the counterterm (54). To be explicit, we compare the diagonal components at zero spatial momentum

$$\Pi_{\text{cl}}^{ii}(q_0, \vec{q} = 0) = 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}|^2 = 0.26 g^2 N T a^{-1} , \quad (55)$$

$$\Pi_{\text{app}}^{ii}(q_0, \vec{q} = 0) = 2g^2 N T a^{-1} \int \frac{d^3 \hat{p}}{(2\pi)^3} \hat{\Omega}_{\vec{p}}^{-2} |\vec{v}_{\text{lat}}| = 0.34 g^2 N T a^{-1} . \quad (56)$$

Comparing the difference between (45) and (46) with the HTL self-energy at zero spatial momentum  $\Pi_{HTL}^{ii}(q_0, \vec{q} = 0) = 3\omega_{\text{pl}}^2 = g^2 T^2/3$ , we obtain an estimate for the maximal error of about 25% for  $a^{-1} = T/\hbar$ . However, the semi-hard modes that give the  $\mathcal{O}(1)$  correction have space-like momenta  $q_0 < |\vec{q}|$  [14]. For these modes we expect (54) to be a better approximation of the classical self-energy (26).

Besides the mismatch between classical HTL's and the counterterms from (46), the lattice spacing dependence of  $\kappa_2$  depends on the magnitude of the  $\mathcal{O}(1)$  correction from the semi-hard modes. Especially when the soft modes dominate the contribution to  $\kappa_2$  this model is suitable for a calculation of the Chern-Simons diffusion rate.

## V. CONCLUSION

In this paper, we studied the linear divergences in classical  $\text{SU}(N)$  gauge theories at finite temperature. Counterterms for these divergences can be incorporated in an (induced) source. Although the divergences are non-local the equations of motion including these counterterms can be given in a local form by introducing auxiliary fields. In the continuum these auxiliary fields depend only on the direction of the velocity  $\vec{v}$ , whereas on the lattice they necessarily also depend on the magnitude of the velocity. We have presented two lattice models with a bounded energy. The first describes a real-time quantum lattice theory with a small lattice spacing  $a_S$ . The requirement that the energy is bounded presents a lower bound on  $a_S$  given the lattice spacing  $a_L$  of the classical model. It is shown that for very small coupling  $g$  the lattice spacing  $a_S$  can be taken small compared to the inverse temperature, which implies that continuum results can be obtained.

In the second model we argued that the restriction to auxiliary fields depending on the direction of the velocity allows for a reasonable approximation (46) for the calculation of quantities dominated by fields with momenta  $(q_0, q) \sim (g^4 T, g^2 T)$ , such as the Chern-Simons diffusion rate.

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